REGRESSION ESTIMATES OF LINEAR REGRESSION COEFFICIENTS

BU-108-M W. R. Blischke July, 1959

Introduction

Consider the chance variables Y_i $(i=1,\cdots,n)$, related to the variables X_i by the usual linear regression model

$$(1) \qquad Y_i=\alpha+\beta X_i+\epsilon_i \quad .$$

Suppose now that the slope β is itself a chance variable and is functionally related to another variable, say Z_i. Thus, while the experimenter can estimate α and β in (1) (e.g., by the method of least squares), the particular β being estimated depends upon another condition of the experiment, namely the value taken on by the variable Z, which may or may not be under the control of the experimenter. If the experimenter can control and accurately measure Z and can specify the function relating Z and β, a procedure may be devised whereby he may make use of this information in the estimation of the slope in equation (1). In practice such a procedure is useful when, for example, a number of experiments involving the variables X and Y are performed at various levels of Z.

In such a situation we will denote the slope in equation (1) by β_i. In general we have, then, that $\beta_i=f(Z_i)$, say, and write

$$(2) \qquad Y_{ij}=\alpha+f(Z_i)\cdot X_{ij}+\epsilon_{ij}$$

where $j=1,\cdots,n_i$, $i=1,\cdots,n$. Estimation procedures will be given for $f(Z_i)$ linear and for $f(Z_i)$ exponential.

I β_i and Z_i Linearly Related

The appropriate model is

$$(3) \qquad Ef(Z_i)=\alpha_0+\beta_0 Z_i \ ,$$

whence (2) becomes

(4) $$Y_{ij} = \alpha+(\alpha_0+\beta_0 Z_i)X_{ij}+\epsilon_{ij}$$

$$= \alpha+\alpha_0 X_{ij}+\beta_0 X_{ij}Z_i+\epsilon_{ij} \quad .$$

Estimation of the parameters α, α_0, and β_0 is easily accomplished by the methods of multiple linear regression. Denote $x_{ij}=X_{ij}-\bar{x}$, $W_{ij}=X_{ij}Z_i$, $w_{ij}=W_{ij}-\bar{w}$, and $\mu=\alpha+\alpha_0\bar{x}+\beta_0\bar{w}$. Then (4) becomes

(5) $$Y_{ij}=\mu+\alpha_0 x_{ij}+\beta_0 w_{ij}+\epsilon_{ij} \quad ,$$

and solution of the least squares equations (cf. Snedecor, 5th Ed., section 14.2) yields the estimators

$$\hat{\mu} = \bar{y},$$

$$\hat{\alpha}_0 = \frac{\Sigma w_{ij}^2 \Sigma x_{ij}y_{ij}-\Sigma x_{ij}w_{ij}\Sigma w_{ij}y_{ij}}{D} \quad ,$$

and $$\hat{\beta}_0 = \frac{\Sigma x_{ij}^2 \Sigma w_{ij}y_{ij}-\Sigma x_{ij}w_{ij}\Sigma x_{ij}y_{ij}}{D} \quad ,$$

where

$$D = \Sigma x_{ij}^2 \Sigma w_{ij}^2-(\Sigma x_{ij}w_{ij})^2 \quad .$$

Variances and covariances of the estimators are estimated as in chapter 14 of Snedecor.

II $f(Z_i)$ Exponential

Suppose that the relationship between Z_i and β_i is expressed by $Ef(Z_i) =\alpha_0 Z_i^{\beta_0}$. Substitution of this expression in (2) yields

(6) $$Y_{ij}=\alpha+\alpha_0 Z_i^{\beta_0} X_{ij}+\epsilon_{ij}$$

and the method of least squares results in a system of non-linear equations in the three unknown parameters, the solution of which is highly non-trivial. Because of the difficulty of these equations, we will treat the relationship between Z_i and β_i as a separate regression problem, use the information on the variables Z_i to construct estimators of the β_i, and substitute these estimates in (2).

To this end the multiplicative model

$$\beta_i = \alpha_0 Z_i^{\beta_0} \delta_i \tag{7}$$

is proposed to represent the relationship between Z_i and β_i. Then a logarithmic transformation yields the linear additive model

$$\log \beta_i = \log \alpha_0 + \beta_0 \log Z_i + \log \delta_i \quad . \tag{8}$$

In order to make use of this expression to include the information about Z in the estimation procedure, preliminary estimates of the β_i are required. We will therefore assume that for each Z_i, β_i is estimated by the use of the relationship between X and Y (and thus that $n_i > 1$ holds for $1 \leq i \leq n$). Denote these preliminary estimates $\hat{\beta}_i$.

Now the methods of linear regression can be applied to the n pairs $(\hat{\beta}_i, Z_i)$ to compute the regression equation

$$\begin{aligned} \widehat{\log \beta_i} &= \widehat{\log \alpha_0} + \hat{\beta}_0 \log Z_i \\ &= \hat{\mu}_{1\cdot 1} + \hat{\beta}_0(\log Z_i - \overline{\log Z}) \end{aligned} \tag{9}$$

where

$$\overline{\log Z} = \frac{\sum_{i=1}^{n} (\log Z_i)}{n}$$

and

$$\hat{\mu}_{1\cdot 1} = \widehat{\log \alpha_0} + \hat{\beta}_0 \overline{\log Z} \quad .$$

For a given Z_i, equation (9) can be used to construct the estimator of β_i.

(10) $$\hat{\beta}_i' = e^{\widehat{\log \beta_i}} = e^{\hat{\mu}_{1\cdot 1} + \hat{\beta}_0(\log Z_i - \overline{\log Z})}$$

Let us assume that $\log \delta_i$ are independent and identically normally distributed with $E \log \delta_i = 0$ and $E(\log \delta_i)^2 = \sigma^2$. Then $\hat{\mu}_{1\cdot 1}$ and $\hat{\beta}_0$ are normally and independently distributed with respective variances σ^2/n and $\sigma^2/\Sigma(\log Z_i - \overline{\log Z})^2 = \sigma^2/S$, say. Thus

$$V(\widehat{\log \beta_i}) = \sigma^2 \left[\frac{1}{n} + \frac{(\log Z_i - \overline{\log Z})^2}{S}\right]$$

for all i. We now proceed to examine the properties of the estimators $\hat{\beta}_i'$ under these assumptions.

<u>Expected value of $\hat{\beta}_i'$</u>. Denote $\bar{z}_G = (Z_1 Z_2 \cdots Z_n)^{\frac{1}{n}}$. Then $\overline{\log Z} = \log \bar{z}_G$. We have

$$\begin{aligned}
E\hat{\beta}_i' &= Ee^{\hat{\mu}_{1\cdot 1}} e^{\hat{\beta}_0(\log Z_i - \overline{\log Z})} \\
&= Ee^{\hat{\mu}_{1\cdot 1}} e^{\hat{\beta}_0(\log Z_i - \log \bar{z}_G)} \\
&= Ee^{\hat{\mu}_{1\cdot 1}} e^{\hat{\beta}_0 \log(Z_i/\bar{z}_G)}
\end{aligned}$$

(11) $$= Ee^{\hat{\mu}_{1\cdot 1}} Ee^{\hat{\beta}_0 \log(Z_i/\bar{z}_G)}$$

since $\hat{\mu}_{1\cdot 1}$ and $\hat{\beta}_0$ are independent. These expectations can be computed by evaluating the moment generating functions of $\hat{\mu}_{1\cdot 1}$ and $\hat{\beta}_0$ at the points t=1 and $t = \log(Z_i/\bar{z}_G)$, respectively.

Now the moment generating function of a normally distributed chance variable V, whose mean and variance are m and σ^2, respectively, is

(12) $$Ee^{Vt} = e^{mt + \frac{1}{2}\sigma^2 t^2}$$

Applying (12) to each term in (11), we obtain

$$Ee^{\hat{\mu}_{1\cdot 1}} = Ee^{\hat{\mu}_{1\cdot 1} t}\Big|_{t=1}$$

$$= e^{E\hat{\mu}_{1\cdot 1} + \frac{1}{2}V(\hat{\mu}_{1\cdot 1})}$$

$$= e^{\hat{\mu}_{1\cdot 1} + \frac{1}{2}\frac{\sigma^2}{n}}$$

and

$$Ee^{\hat{\beta}_0 \log(Z_i/\bar{z}_G)} = Ee^{\hat{\beta}_0 t}\Big|_{t=\log(Z_i/\bar{z}_G)}$$

$$= e^{\beta_0 \log(Z_i/\bar{z}_G) + \frac{\sigma^2}{2S}[\log(Z_i/\bar{z}_G)]^2} .$$

Therefore

$$E\hat{\beta}_i' = e^{\mu_{1\cdot 1} + \beta_0 \log(Z_i/\bar{z}_G) + \frac{\sigma^2}{2}[\frac{1}{n} + \frac{1}{S}\left\{\log(Z_i/\bar{z}_G)\right\}^2]}$$

$$= e^{(\mu_{1\cdot 1} - \beta_0 \log \bar{z}_G) + \beta_0 \log Z_i + \frac{\sigma^2}{2}[\frac{1}{n} + \frac{1}{S}\left\{\log(Z_i/\bar{z}_G)\right\}^2]}$$

$$= \alpha_0 Z_i^{\beta_0} e^{\frac{1}{2}V(\widehat{\log \beta_i})} .$$

Thus $\hat{\beta}_i'$ is not an unbiased estimator of β_i.

<u>Construction of an unbiased estimator and a consistent estimator</u>. If σ^2 is known, then unbiased estimators of the β_i can easily be constructed. For example, the estimators

$$(13) \qquad \hat{\beta}_i^* = \hat{\beta}_i' e^{-\frac{\sigma^2}{2}[\frac{1}{n} + \frac{1}{S}\left\{\log(Z_i/\bar{z}_G)\right\}^2]}$$

are clearly unbiased estimators of the β_i in this case, for

$$E\hat{\beta}_i^* = E\hat{\beta}_i' e^{-\frac{1}{2}V(\widehat{\log \beta_i})}$$

$$= e^{-\frac{1}{2}V(\widehat{\log \beta_i})} E\hat{\beta}_i'$$

$$= \beta_i .$$

If σ^2 is unknown, the problem is no longer trivial. We begin with the estimator of σ^2,

$$s_{1\cdot 1}^2 = \frac{\sum_i [\log \hat{\beta}_i - \overline{\log \hat{\beta}_i} - \hat{\beta}_0(\log Z_i - \log \bar{z}_G)]^2}{n-2} .$$

Since $s_{1\cdot 1}^2$, $\hat{\mu}_{1\cdot 1}$, and $\hat{\beta}_0$ are mutually independent (because of normality), and since $Es_{1\cdot 1}^2 = \sigma^2$, the estimators

$$(14) \qquad \hat{\beta}_i'' = \hat{\beta}_i' \, e^{-\frac{s_{1\cdot 1}^2}{2}\left[\frac{1}{n} + \frac{1}{S}\left\{\log(Z_i/\bar{z}_G)\right\}^2\right]}$$

are suggested as possible candidates for unbiased estimators. We will now show that the $\hat{\beta}_i''$ are actually not unbiased. In a later section we will show that the $\hat{\beta}_i''$ are, however, consistent estimators of the β_i.

<u>Expected value of $\hat{\beta}_i''$</u>. By the independence of $\hat{\mu}_{1\cdot 1}$, $\hat{\beta}_0$, and $s_{1\cdot 1}^2$,

$$E\hat{\beta}_i'' = Ee^{\hat{\mu}_{1\cdot 1} + \hat{\beta}_0(\log Z_i - \log \bar{z}_G) - \frac{1}{2}\cdot s_{1\cdot 1}^2\left[\frac{1}{n} + \frac{1}{S}\left\{\log(Z_i/\bar{z}_G)\right\}^2\right]}$$

$$= Ee^{\hat{\mu}_{1\cdot 1}} \, Ee^{\hat{\beta}_0(\log Z_i - \log \bar{z}_G)} \, Ee^{-\frac{1}{2} s_{1\cdot 1}^2\left[\frac{1}{n} + \frac{1}{S}\left\{\log(Z_i/\bar{z}_G)\right\}^2\right]}$$

To evaluate the last expectation in this expression, we use the fact that the chance variable $(n-2)s_{1\cdot 1}^2$ has a χ^2-distribution with $(n-2)$ degrees of freedom, so that the distribution of $s_{1\cdot 1}^2$ is

$$(15) \qquad f(s_{1\cdot 1}^2) = \frac{1}{\Gamma\left(\frac{n-2}{2}\right)}\left(\frac{n-2}{2\sigma^2}\right)^{\frac{n-2}{2}} (s_{1\cdot 1}^2)^{\frac{n-4}{2}} e^{-\frac{(n-2)s_{1\cdot 1}^2}{2\sigma^2}} .$$

This is a gamma distribution with parameters $(n-2)/2$ and $2\sigma^2/(n-2)$. Its moment generating function is

$$(16) \qquad Ee^{s_{1\cdot 1}^2 t} = \left(\frac{1}{1 - \frac{2\sigma^2}{n-2} t}\right)^{\frac{n-2}{2}}$$

provided that $t < \frac{n-2}{2\sigma^2}$. (See Mood, section 6.3)

Evaluating (16) at $t = -\frac{1}{2}[\frac{1}{n} + \frac{1}{S}\{\log(Z_i/\bar{z}_G)\}^2]$, we obtain the desired expectation

$$Ee^{-\frac{1}{2}s^2_{1\cdot 1}[\frac{1}{n} + \{\log(Z_i/\bar{z}_G)\}^2]} = \left[\frac{1}{1+\frac{\sigma^2}{n-2}[\frac{1}{n} + \frac{1}{S}\{\log(Z_i/\bar{z}_G)\}^2]}\right]^{\frac{n-2}{2}},$$

so that

$$E\hat{\beta}_i'' = \alpha_0 Z_i^{\beta_0} e^{\frac{1}{2}V(\widehat{\log \beta_i})} [1+\frac{1}{n-2}V(\widehat{\log \beta_i})] .$$

Thus $\hat{\beta}_i''$ is also biased. But since

$$e^x = \lim_{r\to\infty} (1+\frac{x}{r})^r ,$$

we have

$$\lim_{\frac{n-2}{2}\to\infty} \left[\frac{1}{1+\frac{\sigma^2}{n-2}[\frac{1}{n} + \frac{1}{S}\{\log(Z_i/\bar{z}_G)\}^2]}\right]^{\frac{n-2}{2}}$$

$$= \lim_{\frac{n-2}{2}\to\infty} \left[\frac{1}{1+\frac{2}{n-2}\cdot\frac{\sigma^2}{2}[\frac{1}{n} + \frac{1}{S}\{\log(Z_i/\bar{z}_G)\}^2]}\right]^{\frac{n-2}{2}}$$

$$= e^{-\frac{\sigma^2}{2}[\frac{1}{n} + \frac{1}{S}\{\log(Z_i/\bar{z}_G)\}^2]} ,$$

so that for large n,

$$E\hat{\beta}_i'' = \alpha_0 Z_i^{\beta_0} e^{\frac{1}{2}V(\widehat{\log \beta_i})} [1+\frac{1}{n-2} V(\widehat{\log \beta_i})]^{-\frac{n-2}{2}}$$

$$\doteq \alpha_0 Z_i^{\beta_0} e^{\frac{1}{2}V(\widehat{\log \beta_i})} e^{-\frac{1}{2}V(\widehat{\log \beta_i})}$$

$$= \alpha_0 Z_i^{\beta_0}$$

$$= \beta_i ,$$

i.e., as $n \to \infty$, $E\hat{\beta}_i'' \to \beta_i$. Thus $\hat{\beta}_i''$ is asymptotically unbiased.

Variances of the estimators. 1. $V(\hat{\beta}_i')$.

$$E(\hat{\beta}_i')^2 = E\left\{ e^{\hat{\mu}_{1\cdot 1}} e^{\hat{\beta}_0 \log(Z_i/\bar{z}_G)} \right\}^2$$

$$= Ee^{2\hat{\mu}_{1\cdot 1}} e^{2\hat{\beta}_0 \log(Z_i/\bar{z}_G)}$$

$$= \left(Ee^{t\hat{\mu}_{1\cdot 1}}\Big|_{t=2} \right) \left(Ee^{\hat{\beta}_0 t}\Big|_{t=2 \log(Z_i/\bar{z}_G)} \right)$$

$$= e^{2\mu_{1\cdot 1} + \frac{2\sigma^2}{n}} e^{2\beta_0 \log(Z_i/\bar{z}_G) + 2\left\{ \log(Z_i/\bar{z}_G) \right\}^2 \frac{\sigma^2}{S}}$$

$$= e^{2[\hat{\mu}_{1\cdot 1} + \beta_0 \log(Z_i/\bar{z}_G) + \sigma^2(\frac{1}{n} + \frac{1}{S}\left\{ \log(Z_i/\bar{z}_G) \right\}^2)]}$$

$$= [e^{\hat{\mu}_{1\cdot 1} - \beta_0 \log \bar{z}_G + \beta_0 \log Z_i + V(\widehat{\log \beta_i})}]^2$$

$$= [\alpha_0 Z_i^{\beta_0} e^{V(\widehat{\log \beta_i})}]^2 \quad .$$

Thus

$$V(\hat{\beta}_i') = \alpha_0^2 Z_i^{2\beta_0} e^{2V(\widehat{\log \beta_i})} - [\alpha_0 Z_i^{\beta_0} e^{\frac{1}{2}V(\widehat{\log \beta_i})}]^2$$

$$= (\alpha_0 Z_i^{\beta_0})^2 e^{V(\widehat{\log \beta_i})} [e^{V(\widehat{\log \beta_i})} - 1] \quad .$$

2. $V(\hat{\beta}_i'')$

$$E(\hat{\beta}_i'')^2 = Ee^{2\hat{\mu}_{1\cdot 1} + 2\hat{\beta}_0 \log(Z_i/\bar{z}_G) - s^2_{1\cdot 1}[\frac{1}{n} + \frac{1}{S}\left\{ \log(Z_i/\bar{z}_G) \right\}^2]}$$

$$= Ee^{t\hat{\mu}_{1\cdot 1}}\Big|_{t=2} \; Ee^{\hat{\beta}_0 t}\Big|_{t=2 \log(Z_i/\bar{z}_G)} \; Ee^{s^2_{1\cdot 1} t}\Big|_{t=-\left[\frac{1}{n} + \frac{1}{S}\left\{ \log(Z_i/\bar{z}_G) \right\}^2\right]}$$

$$= e^{2\mu_{1\cdot 1}+\frac{2\sigma^2}{n}} e^{2\beta_0 \log(Z_i/\bar{z}_G)+2\beta_0[\log(Z_i/\bar{z}_G)]^2} \left\{ 1+\frac{2\sigma^2}{n-2}\left[\frac{1}{n} + \frac{1}{S}\left\{\log(Z_i/\bar{z}_G)\right\}^2\right]\right\}^{-\frac{n-2}{2}}$$

$$= \alpha_0^2 Z_i^{2\beta_0} e^{2V(\widehat{\log \beta_i})} \left\{ 1+\frac{2\sigma^2}{n-2}\left[\frac{1}{n}+\frac{1}{S}\left\{\log\ (Z_i/\bar{z}_G)\right\}^2\right]^{-\frac{n-2}{2}}\right\}$$

$$\doteq (\alpha_0 Z_i^{\beta_0})^2 e^{2V(\widehat{\log \beta_i})} e^{-V(\widehat{\log \beta_i})} \qquad \text{for n large}$$

$$= (\alpha_0 Z_i^{\beta_0})^2 e^{V(\widehat{\log \beta_i})} \quad .$$

Thus

$$V(\hat{\beta}_i'') = (\alpha_0 Z_i^{\beta_0})^2 e^{2V(\widehat{\log \beta_i})} \left[1+\frac{2\sigma^2}{n-2}\left\{\frac{1}{n}+\frac{1}{S}(\log(Z_i/\bar{z}_G))^2\right\}\right]^{-\frac{n-2}{2}}$$

$$- \left\{ \frac{\alpha_0 Z_i^{\beta_0} e^{\frac{1}{2}V(\widehat{\log \beta_i})}}{[1+\frac{1}{n-2}V(\widehat{\log \beta_i})]^{\frac{n-2}{2}}} \right\}^2$$

$$= (\alpha_0 Z_i^{\beta_0})^2 e^{V(\widehat{\log \beta_i})} \left\{ [1+\frac{2}{n-2}V(\widehat{\log \beta_i})]^{-\frac{n-2}{2}} e^{V(\widehat{\log \beta_i})} - [1+\frac{1}{n-2}\ V(\widehat{\log \beta_i})\ ]^{-\frac{n-2}{2}} \right\}$$

$$\doteq (\alpha_0 Z_i^{\beta_0})^2 (e^{V(\widehat{\log \beta_i})} -1) \quad .$$

<u>Proof of consistency of $\hat{\beta}_i''$</u>. We have shown that $E\hat{\beta}_i'' \to \beta_i$ as $n \to \infty$. By a theorem of Wilks (Theorem A, p. 134), this condition along with the condition, $V(\hat{\beta}_i'') \to 0$ an $n \to \infty$, are sufficient conditions that $\hat{\beta}_i''$ be a consistent estimator of β_i. Thus to prove consistency, it is sufficient to show that $V(\hat{\beta}_i'') \to 0$ as $n \to \infty$.

We have

$$V(\hat{\beta}_i'')=(\alpha_0 Z_i^{\beta_0})^2 e^{V(\widehat{\log \beta_i})}\left\{[1+\frac{2}{n-2}V(\widehat{\log \beta_i})]^{-\frac{n-2}{2}} e^{V(\widehat{\log \beta_i})} - [1+\frac{1}{n-2}V(\widehat{\log \beta_i})]^{-\frac{n-2}{2}}\right\}$$

Now

$$\lim_{n\to\infty} e^{V(\widehat{\log \beta_i})} = \lim_{n\to\infty} e^{\sigma^2[\frac{1}{n}+\left\{\log(Z_i/\bar{z}_G)\right\}^2/\sum_{i=1}^{n}(\log Z_i-\log \bar{z}_G)^2]}$$
$$= 1$$

since as $n\to\infty$, $\frac{1}{n}\to 0$ and under very general conditions $k/\sum_{i=1}^{n}(\log Z_i-\log \bar{z}_G)^2\to 0$ as well.

Thus also

$$\lim_{n\to\infty}[1+\frac{2}{n-2}V(\widehat{\log \beta_i})]^{-\frac{n-2}{2}} = \lim_{n\to\infty} e^{-V(\widehat{\log \beta_i})}$$
$$= 1$$
$$= \lim_{n\to\infty}[1+\frac{1}{n-2}V(\widehat{\log \beta_i})]^{-\frac{n-2}{2}}$$

Therefore

$$\lim_{n\to\infty}[V(\hat{\beta}_i'')] = (\alpha_0 Z_i^{\beta_0})^2\,(1)\,[(1)(1)-1] = 0$$

Hence $\hat{\beta}_i''$ is consistent.

<u>Estimation of the variances</u>. Estimates of $V(\hat{\beta}_i')$ and $V(\hat{\beta}_i'')$ can be constructed by the same procedure used to arrive at the estimators $\hat{\beta}_i''$. Namely:

1. $\hat{V}(\hat{\beta}_i')$. The quantity $(\hat{\beta}_i')^2-(\hat{\beta}_i'')^2$ is an estimator of $E(\hat{\beta}_i')^2-(E\hat{\beta}_i')^2$, for $(\hat{\beta}_i')^2$ is an unbiased estimator of $E(\hat{\beta}_i')^2$ and for large n

$$E(\hat{\beta}_i'')^2 \approx (\alpha_0 Z_i^{\beta_0})^2 e^{V(\widehat{\log \beta_i})} = (E\,\hat{\beta}_i')^2$$

This estimator is consistent (but, it must be pointed out, is biased downward).

2. $\hat{V}(\hat{\beta}_i'')$. Let

$$H = \hat{\beta}_i' \, e^{-s^2_{1\cdot1}\left[\frac{1}{n} + \frac{1}{S}\left\{\log(Z_i/\bar{z}_G)\right\}^2\right]}$$

$$= \hat{\beta}_i'' \, e^{-\frac{1}{2} s^2_{1\cdot1}\left[\frac{1}{n} + \frac{1}{S}\left\{\log(Z_i/\bar{z}_G)\right\}^2\right]}$$

Then, as above,

$$E(H^2) = Ee^{2\hat{\mu}_{1\cdot1}+2\hat{\beta}_0 \log(Z_i/\bar{z}_G)-2s^2_{1\cdot1}\left[\frac{1}{n} + \frac{1}{S}\left\{\log(Z_i/\bar{z}_G)\right\}^2\right]}$$

$$= (\alpha_0 Z_i^{\beta_0})^2 \, e^{2V(\widehat{\log \beta_i})}\left\{\frac{1}{1+\frac{2\sigma^2}{n-2}t}\right\}^{\frac{n-2}{2}}\Bigg|_{t=-2\left[\frac{1}{n} + \frac{1}{S}\log(Z_i/\bar{z}_G)^2\right]}$$

$$= (\alpha_0 Z_i^{\beta_0})^2 \, e^{2V(\widehat{\log \beta_i})}\left\{\frac{1}{1+\left(\frac{2}{n-2}\right)2V(\widehat{\log \beta_i})}\right\}^{\frac{n-2}{2}}$$

$$\doteq (\alpha_0 Z_i^{\beta_0})^2$$

Thus a consistent estimator of $V(\hat{\beta}_i'')$ is $(\hat{\beta}_i'')^2 - H^2$.

<u>Discussion</u>

Many aspects of this problem have been left untouched. A few of the unanswered questions are the following:

1. It can easily be shown that $\hat{\beta}_i'$ is itself a consistent estimator. Can it be proved that $\hat{\beta}_i''$ converges (stochastically) to β_i more rapidly? What are the M.S.E.'s of the two estimators?

2. What do the assumptions made on $\log \delta_i$ imply about the ϵ_{ij}? What $\hat{\beta}_i$ should be used?

3. Suppose the ϵ_{ij} are normally distributed. What does this imply about the distribution of $\log \delta_i$? Can the problem be solved in this case?

4. In the present problem is there an unbiased estimator when σ^2 is unknown?

5. The problem can be considered for many other functions $f(Z_i)$.

6. Many other conditions of the experiment can vary. What if, for example, $V(Y_{ij})=g(Z_i)$, or $V(Y_{ij})=h(X_{ij}Z_i)$, etc.?

References

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